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ON MULTISTEP APPROXIMATION OF SEMIGROUPS IN BANACH SPACES

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ON MULTISTEP APPROXIMATION OF SEMIGROUPS IN BANACH SPACES

APPROXIMATION DE SEMI-GROUPES DANS DES ESPACES DE BANACH PAR DES METHODES MULTIPAS

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Abstract : We consider a multistep rational approximation of a bounded, strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of A-stability assumption.

Résumé : Etant donné un semi-groupe fortement continu dans un espace de Banach et son approximation par une méthode rationnelle multipas, nous étudions la convergence sous une hypothèse de A-stabilité affaiblie.

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ON MULTISTEP APPROXIMATION OF SEMIGROUPS IN BANACH SPACES

by Michel Crouzeix

Abstract : we consider a multistep rational approximation of a bounded strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of A-stability assumption.

1. Introduction.

Given a uniformly bounded, strongly continuous semigroup e^{tA} on a Banach space X :

$$\| e^{tA} \| \le C_0 \quad , \tag{1}$$

and u_0 in X, we approximate the value $u(t) = e^{tA} u_0$ at the time $t_n = n \Delta t$ by the solution u_n of the q-step method

$$u_{n+1} = \sum_{j=0}^{q-1} r_j(\Delta t A)\, u_{n-j} \quad , n \ge q-1 \, , \tag{2}$$

starting from the procedure

$$u_j = d_j(\Delta t A)\, u_0 \, , \quad j=1,\dots,q-1 \tag{3}$$

where $r_j(z), d_j(z)$ are rational functions uniformly bounded for $\mathrm{Re}\, z \le 0$ and $\Delta t > 0$ denotes the time step.

We first consider the scalar case where $X = \mathbf{C}$, $u_0 = 1$ and $A = \lambda \in \mathbf{C}$ with $\mathrm{Re}\, \lambda \le 0$. In order to obtain a global error $u(t_n) - u_n = O(\Delta t^p)$, it is natural to assume a local error in (2) and in (3) of order p+1, that is to say, setting $z = \lambda \Delta t$,

$$e^{qz} - \sum_{j=0}^{q-1} r_j(z)\, e^{(q-j-1)z} = O(z^{p+1}) \tag{4}$$

and

$$e^{jz} - d_j(z) = O(z^{p+1}) \, , \quad j=1,\dots,q-1. \tag{5}$$

We introduce the polynomial

$$P(X ; z) = X^q - \sum_{j=0}^{q-1} r_j(z)\, X^{q-j-1} \, , \tag{6}$$

associated with the linear recursion formula (2) ; then Condition (4) may be rewritten as

$$P(e^z ; z) = O(z^{p+1}). \tag{7}$$

We shall say that Scheme (2) is A-stable with defect $k \geq 0$ if :
for all z *with* $\mathrm{Re}\, z \leq 0$, *the roots of* $P(\,.\,;z)$ *lie in the unit disk and the mulplicities of the unimodular roots are less than* $k+1$.

The integer k is chosen as small as possible. We remark that, from (7), 1 is a root of $P(\,.\,;0)$. The case $k = 0$ corresponds to the classical notion of A-stability.

Remark : With each monic polynomial $P(\,.\,;z)$, whose coefficients are rational functions of z, we can associate a method of Form (2) ; so, we can construct high order schemes in a simple way :
- if the method associated with the polynomial $P(\,.\,;z)$ is of order p and A-stable with defect k, the method associated with $P(\,.\,;z)^m$ is of order $mp+m-1$ and A-stable with defect $km+m-1$. Similarly, if another method, associated with $Q(\,.\,;z)$, is of order q and A-stable with defect r, the method associated with $P(\,.\,;z)Q(\,.\,;z)$ is of order $p+q+1$ and A-stable with defect $k+r+1$.

The idea to consider this kind of construction was born during some discussions with G.A. Baker, see also [1].

In this note we prove the following theorem :

Theorem 1. *If Scheme* (2) *is A-stable with defect* k , *if Conditions* (5) *and* (7) *are satisfied and if* u_0 *belongs to* $D(A^{p+1})$, *then the following inequality holds*

$$\| u(t_n) - u_n \|_X \leq C_1\, C_0\, t_n^{k+1}\, \Delta t^{p-k}\, \| A^{p+1} u_0 \|_X \quad ,$$

where the constant C_1 *depends only on the rational functions* r_j *and* d_j .

The proof of this theorem uses the mathematical framework described in the paper of Brenner and Thomée [4] and some technical lemmas ; the case of analytical semigroups is considered in Section 3 and the hilbertian case in Section 4. For related works, see [2], [3], [6], [7].

2. Proof of Theorem 1.

We introduce the Frobenius matrix R(z) associated with $P(\,.\,;z)$ and the corresponding linear operator $R(\Delta t A)$ in $\mathcal{L}(X^q,X^q)$ defined by

$$R(z) = \begin{pmatrix} r_0 & r_1 & \cdots & r_{q-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \quad 0 \end{pmatrix} \qquad R(\Delta t A) = \begin{pmatrix} R_0 & R_1 & \cdots & R_{q-1} \\ I & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \quad 0 \end{pmatrix}$$

where r_i stands for $r_i(z)$ and R_i for $r_i(\Delta t A)$; we also consider the vectors in X^q of approximate and exact solutions :

$$U_n = \begin{pmatrix} u_{n+q-1} \\ u_{n+q-2} \\ \vdots \\ u_n \end{pmatrix}, V_n = \begin{pmatrix} u(t_{n+q-1}) \\ u(t_{n+q-2}) \\ \vdots \\ u(t_n) \end{pmatrix}, U_0 = \begin{pmatrix} d_{q-1}(\Delta tA)u_0 \\ \vdots \\ d_1(\Delta tA)u_0 \\ u_0 \end{pmatrix}, V_0 = \begin{pmatrix} e^{(q-1)\Delta tA}u_0 \\ \vdots \\ e^{\Delta tA}u_0 \\ u_0 \end{pmatrix}.$$

Clearly (2) implies

$$U_{n+1} = R(\Delta tA)\, U_n \ , \ V_{n+1} = e^{\Delta tA} V_n \ ;$$

therefore

$$U_n = R(\Delta tA)^n U_0, \quad V_n = e^{n\Delta tA} V_0$$

and we have the error representation formula

$$U_n - V_n = H_n(\Delta tA)\, u_0 + G_n(\Delta tA)\, u_0 \ , \qquad (8)$$

with

$$H_n(z) = (R(z)^n - e^{nz} I) \begin{pmatrix} e^{(q-1)z} \\ \vdots \\ e^z \\ 1 \end{pmatrix}, \ G_n(z) = R(z)^n \begin{pmatrix} d_{q-1}(z) - e^{(q-1)z} \\ \vdots \\ d_1(z) - e^z \\ 0 \end{pmatrix} . \qquad (9)$$

Lemma 2. a) *If z_0 is not a pole of $R(z)$, there exist a neighborhood $\mathcal{V}_0$ of z_0 in $\bar{\mathbb{C}}$ and a constant C such that*

$$\forall\, n \geq k_0, \forall\, z \in \mathcal{V}_0 \ , \quad \|R(z)^n\| \leq C \left[\rho(R(z))^n + n^{k_0} \rho(R(z))^{n-k_0} \right] ,$$

where $\rho(R(z))$ denotes the spectral radius of $R(z)$ and (k_0+1) the highest multiplicity of the eigenvalues of $R(z_0)$ having modulus $\rho(R(z_0))$.

b) *If furthermore the eigenvalues of $R(z_0)$ which have a modulus equal to $\rho(R(z_0))$ are differentiable over $\mathcal{V}_0$, there exists a constant C such that*

$$\forall\, n \geq k_0, \forall\, z \in \mathcal{V}_0 \ , \ \left\| \frac{d}{dz}(R(z)^n) \right\| \leq C \left[n\, \rho(R(z))^{n-1} + n^{k_0+1} \rho(R(z))^{n-k_0-1} \right].$$

Sketch of proof· a) In a neighborhood of z_0, it is possible to find an analytic and invertible matrix $H(z)$ such that

$$H^{-1}(z)\, R(z)\, H(z) = \mathrm{diag}(\, R_s(z)\,) \qquad (10)$$

is a block diagonal matrix where each square matrix $R_s(z)$ corresponds to one of the distinct eingenvalues of $R(z_0)$. Clearly $\|R(z)^n\| \leq \max_s \|R_s(z)^n\|$;

since the matrix $R(z)$ is not derogatory and the eigenvalues of $R(z_0)$ are continuous at the point z_0, we can find invertible matrices $P_s(z)$, continuous at the point z_0, such that

$$P_s(z)^{-1}\, R_s(z)\, P_s(z) = \begin{pmatrix} \lambda_{s0} & & & 0 \\ 1 & \lambda_{s1} & & \\ & 1 & \ddots & \\ & & \ddots & \lambda_{sks} \\ 0 & & 1 & \end{pmatrix} = J_s(z) \ ,$$

where λ_{sj} stands for $\lambda_{sj}(z)$. Part a) follows from inequalities

$$\| J_s(z)^n \| \le \rho(J_s(z))^n + n\,\rho(J_s(z))^{n-1} + \ldots\ldots + \binom{k_s}{n}\,\rho(J_s(z))^{n-k_s}$$

and

$$\| R_s(z)^n \| \le C \, \| J_s(z)^n \| .$$

b) From (10), we deduce

$$\| \frac{d}{dz}(R(z))^n \| \le C \max_s \left(\| R_s(z)^n \| + \| \frac{d}{dz}(R_s(z))^n \| \right) ;$$

the only difficulty is to obtain an estimate of $\| \frac{d}{dz}(R_s(z))^n \|$ when

$\rho(R_s(z_0)) = \rho(R(z_0))$. But, in this case the matrix $P_s(z)$ may be chosen differentiable and therefore

$$\| \frac{d}{dz}(R_s(z))^n \| \le C \left(\| J_s(z)^n \| + \| \frac{d}{dz}(J_s(z))^n \| \right) ;$$

for the matrix $J_s(z)^n$ we have the following upper bound

$$\| \frac{d}{dz}(J_s(z))^n \| \le C \left[n\,\rho(J_s(z))^{n-1} + \ldots + \binom{k_s}{n}\,(n - k_s - 1)\,\rho(J_s(z))^{n-k_s-1} \right] .$$

Lemma 3. *If Scheme* (2) *is A-stable with defect* k, *the unimodular eigenvalues of* R(iy) , *when* y *is real, are twice differentiable near* y.

Proof : First, we remark that the eigenvalues of R(z) are the roots of the polynomial P(. ; z). Without loss of generality, we give the proof only for the case $y = 0$ and for the eigenvalues which converge to 1 as $z \to 0$. These eigenvalues can be written as Puiseux series

$$\lambda_j(z) = 1 + a_j z^{r_j} + b_j z^{s_j} + o(z^{s_j}) ,$$

where $0 < r_j < s_j$ are rational and $a_j \neq 0$. Since $|\lambda_j(z)| \le 1$ when $\mathrm{Re}\, z \le 0$,we cannot have $r_j > 1$; neither can $r_j = p_j / q_j$ be < 1, since each determination of z^{1/q_j} has to be considered. Therefore we have $r_j = 1$ and $a_j > 0$. Considering the case $z = iy$,we must have $0 \le 1 - \mathrm{Re}\,\lambda_j(iy) \approx - \mathrm{Re}(b_j (iy)^{s_j})$, which implies $s_j \ge 2$.

Using the compactness of $\overline{\mathbf{R}}$ and the previous lemma, we obtain the following corollary.

Corollary 4. *If the scheme is A-stable with defect* k , *there exists a constant* C *such that*

$$\| R(iy)^n \| \le C\, n^k ,$$

$$\forall\, n \ge 1 , \forall\, y \in \mathbf{R} ,$$

$$\| \frac{d}{dy} R(iy)^n \| \le C\, n^{k-1} .$$

Now we resume the study of the convergence estimates . Since

$$H_1(z) = (R(z) - e^z I)(e^{(q-1)z},\ldots\ldots,e^z,1)^T = (P(e^z;z),0,\ldots\ldots,0)^T = O(z^{p+1}) ,$$

we obtain from (9) and (7)

$$H_n(z) = \sum_{j=0}^{n-1} e^{jz} R(z)^{n-j-1} H_1(z) = O(z^{p+1}).$$

Similarly, using (5) , we have $G_n(z) = O(z^{p+1})$.
Therefore the functions

$$\tilde{H}_n(z) = H_n(z)/z^{p+1} \quad , \quad \tilde{G}_n(z) = G_n(z)/z^{p+1} \tag{11}$$

are analytic and uniformly bounded in the halfplane $Re z \le 0$.
Using the background described in Brenner-Thomée, there exist two functions $\mathcal{H}_n$ and $\mathcal{G}_n$ in $L^1(\mathbf{R})^q$ such that, for $Re z \le 0$,

$$\tilde{H}_n(z) = \int_0^\infty e^{tz}\, \mathcal{H}_n(t)\, dt \ , \quad \tilde{G}_n(z) = \int_0^\infty e^{tz}\, \mathcal{G}_n(t)\, dt \ , \tag{12}$$

and

$$\mathcal{H}_n(t) = \mathcal{G}_n(t) = 0 \text{ for } t < 0 . \tag{13}$$

We define the two operators in $\mathcal{L}(X^q,X^q)$

$$\tilde{H}_n(\Delta tA) = \int_0^\infty e^{t\Delta tA}\, \mathcal{H}_n(t)\, dt \ , \quad \tilde{G}_n(\Delta tA) = \int_0^\infty e^{t\Delta tA}\, \mathcal{G}_n(t)\, dt, \tag{14}$$

and we have from (8) and (11) , if u_0 is in $D(A^{p+1})$,

$$U_n - V_n = \Delta t^{p+1}\, \tilde{H}_n(\Delta tA)\, A^{p+1} u_0 + \Delta t^{p+1}\, \tilde{G}_n(\Delta tA)\, A^{p+1} u_0 .$$

Now, Theorem 1 is a simple consequence of the following lemma :

Lemma 5 . *Under the hypotheses of Theorem 1, there exists a constant* C *such that*

$$\| \tilde{H}_n(\Delta tA) \| \le C\, C_0\, n^{k+1} \quad \text{and} \quad \| \tilde{G}_n(\Delta tA) \| \le C\, C_0\, n^{k+1/2} .$$

Proof . We have, from (14) and (1),

$$\| \tilde{H}_n(\Delta tA) \| \le C_0 \| \mathcal{H}_n \|_{L^1(\mathbf{R})^q} \text{ and } \| \tilde{G}_n(\Delta tA) \| \le C_0 \| \mathcal{G}_n \|_{L^1(\mathbf{R})^q} ,$$

therefore it is sufficient to prove that

$$\| \mathcal{H}_n \|_{L^1(\mathbf{R})^q} \le C\, n^{k+1} \quad \text{and} \quad \| \mathcal{G}_n \|_{L^1(\mathbf{R})^q} \le C\, n^{k+1/2} . \tag{15}$$

Let us introduce the functions : $\hat{\mathcal{H}}_n(y) = H_n(iy)$, $\hat{\mathcal{G}}_n(y) = G_n(iy)$;
from (12) and (13), $\hat{\mathcal{H}}_n$ and $\hat{\mathcal{G}}_n$ are the Fourier transforms of $\mathcal{H}_n$ and $\mathcal{G}_n$.
A Carlson 's inequality gives

$$\| \mathcal{G}_n \|_{L^1(\mathbf{R})^q} \le C \| \hat{\mathcal{G}}_n \|_{L^2(\mathbf{R})^q}^{1/2} \| \hat{\mathcal{G}}_n' \|_{L^2(\mathbf{R})^q}^{1/2} ;$$

from Corollary 4, we have

$$\| \hat{\mathcal{G}}_n(y) \| \le C\, n^k \min \left(\frac{1}{|y|^{p+1}} , 1 \right)$$

and

$$\| \hat{\mathcal{G}}_n'(y) \| \le C\, n^{k+1} \min \left(\frac{1}{|y|^{p+1}} , 1 \right)$$

therefore

$$\| \hat{\mathcal{G}}_n \|_{L^2(\mathbf{R})^q} \le C\, n^k , \quad \| \hat{\mathcal{G}}_n' \|_{L^2(\mathbf{R})^q} \le C\, n^{k+1} ,$$

which shows the second inequality in (15).

In order to prove the first inequality, we introduce the notations

$$S(z) = \begin{pmatrix} s_0 & s_1 & \cdots & s_{q-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad D(z) = \begin{pmatrix} e^{(q-1)z} & & & 0 \\ & \ddots & & \\ & & e^{z} & \\ 0 & & & 1 \end{pmatrix}$$

where s_j stands for $s_j(z) = r_j(z)\, e^{-(j-1)z}$. We have
$R(z) = e^z\, D(z)\, S(z)\, D(z)^{-1}$, therefore

$$\hat{\mathcal{H}}_n(y) = \frac{1}{(iy)^{p+1}} e^{niy}\, D(iy)\, (S(iy)^n - I)E = e^{niy}\, D(iy)\, \hat{T}_n(y) \qquad (16)$$

where $E=(1,1,\dots,1)^T$ and

$$\hat{T}_n(y) = \frac{1}{(iy)^{p+1}} (S(iy)^n - I)\, E . \qquad (17)$$

We remark that $\hat{T}_n$ is the Fourier transform of T_n which, from (16), is related to $\mathcal{H}_n$ by the relation , (on the j^{th} component),

$$\mathcal{H}_n(t)_j = T_n(t-(n+q-j))_j$$

therefore , using a Carlson's inequality , we obtain

$$\| \mathcal{H}_n \|_{L^1(\mathbf{R})^q} \le C' \| T_n \|_{L^1(\mathbf{R})^q} \le C \| \hat{T}_n \|_{L^2(\mathbf{R})^q}^{1/2} \| \hat{T}_n' \|_{L^2(\mathbf{R})^q}^{1/2} .$$

The following lemma completes the proof of the first inequality in (15).

LEMMA 6 . *If the multiplicity of the root* 1 *of* P (. ; 0) *is* k+1, *then we have*

$$\| \hat{T}_n \|_{L^2(\mathbf{R})} \le C\, n^{k+1-1/(2r+2)} \quad \text{and} \quad \| \hat{T}_n' \|_{L^2(\mathbf{R})} \le C\, n^{k+1+1/(2r+2)},$$

where r *is a rational (which will be defined in the proof).*
If the multiplicity is less than or equal to k, *then*

$$\| \hat{T}_n \|_{L^2(\mathbf{R})} \le C\, n^{k} \quad \text{and} \quad \| \hat{T}_n' \|_{L^2(\mathbf{R})} \le C\, n^{k+1}.$$

Proof . We consider only the case where the multiplicity of the root 1 is k+1, the other case is easier . Relation (17) and Corollary 4 imply

$$\| \hat{T}_n(y) \| \le C\, n^k / |y|^{p+1} \quad \text{and} \quad \| \hat{T}_n'(y) \| \le C(n^k / |y|^{p+2} + n^{k+1} / |y|^{p+1}). \quad (18)$$

In the definition of S(z) , we remark that $\mu(z)$ is an eigenvalue of S(z) if and only if $\lambda(z) = e^z\, \mu(z)$ is an eigenvalue of R(z) , and that the algebraic multiplicity is preserved. From Lemma 2, we can find, in a neiborhood $\mathcal{V}$ of zero on the imaginary axis, an invertible and twice differentiable matrix H(z) such that

$$\forall\, z \in \mathcal{V},\ S(z) = H(z)^{-1} \begin{pmatrix} A(z) & 0 \\ 0 & B(z) \end{pmatrix} H(z)\ , \qquad (19)$$

where the matrix B(z) - I is invertible and the (k+1)x(k+1) matrix A(z) has the form

$$A(z) = \begin{pmatrix} \mu_0(z) & & & \\ 1 & \mu_1(z) & 0 & \\ & 1 & \ddots & \\ & & \ddots & \mu_k(z) \\ 0 & & & 1 \end{pmatrix}\ , \quad \text{with} \quad \mu_j(0) = 1\ .$$

Now we introduce the twice differentiable vectors $A_n(z)$, $B_n(z)$, a(z), b(z), ß(z) by

$$\begin{pmatrix} A_n(z) \\ B_n(z) \end{pmatrix} = H(z)\, \hat{T}_n(z)\ , \qquad \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = H(z)\, E\ , \quad ß(z) = \frac{1}{z^{p+1}} b(z)\ ; \qquad (20)$$

then

$$A_n(z) = \frac{1}{z^{p+1}} \left(A(z)^n - I \right) a(z)\ , \quad B_n(z) = \left(B(z)^n - I \right) ß(z)\ . \qquad (21)$$

Since $ß(z) = (B(z) - I)^{-1} B_1(z)$ is twice differentiable, we obtain

$$\| B_n(iy) \| \le C\, n^k\ , \quad \| B_n'(iy) \| \le C\, n^{k+1}\ , \quad \text{for } iy \in \mathcal{V}\ . \qquad (22)$$

We assume that the order of the eigenvalues $\mu_j(z)$ of $A(z)$ has been chosen in order to ensure

$$0 \le r = r_0 \le r_1 \le \ldots \le r_k ,$$

for the first exponent of the Puiseux expansion

$$\mu_j(z) = 1 + a_j z^{r_j+1} + \ldots\ldots , \quad r_j \in \mathbf{Q} , \quad a_j \neq 0 .$$

Noticing that

$$A_n(z) = \sum_{j=0}^{n-1} A(z)^j A_1(z) \quad \text{and} \quad \| A'(z) \| \le C |z|^r ,$$

we get

$$\| A_n(z) \| \le C n^{k+1} , \quad \| A_n'(z) \| \le C n^{k+1} (1 + n |z|^r) . \tag{23}$$

With the notations

$$a(z) = (a_0(z), \ldots, a_k(z))^T , \quad e_0 = (1,0,\ldots,0)^T$$

$$c(z) = (A(z) - I)(a(z) - a_0(z) e_0) = (0, c_1(z), \ldots, c_k(z))^T , \tag{24}$$

$$(c(z) = 0 \quad \text{when } k = 0),$$

we can also write

$$A_n(z) = \frac{1}{z^{p+1}} \left[a_0(z) (A(z)^n - I) e_0 + \sum_{j=0}^{n-1} A(z)^j c(z) \right] . \tag{25}$$

Since the first row and the first column of $A(z)$ are not used in $A(z)^j c(z)$, this term is bounded by $C j^{k-1} \|c(z)\|$. Then

$$\| A_n(z) \| \le C n^k (|a_0(z)| + \|c(z)\|) / |z|^{p+1} . \tag{26}$$

Similarly, by derivation of (25), we obtain

$$\| A_n'(z) \| \le (p+1) \|A_n(z)\|/|z| + C n^k (|a_0'(z)| + \|c'(z)\|) / |z|^{p+1} + C n^{k+1} |z|^r (|a_0(z)| + \|c(z)\|) / |z|^{p+1} . \tag{27}$$

From the relations

$$(A(z) - I) a(z) = z^{p+1} A_1(z) ,$$

$$(\mu_0(z) - 1) a_0(z) = z^{p+1} (A_1(z))_0 ,$$

we get

$$a_0(z) = O(z^{p-r}) , \quad a_0'(z) = O(z^{p-r-1}) ,$$

$$c(z) = z^{p+1} A_1(z) - a_0(z) (A(z) - I) e_0 = O(z^{p-r}) ,$$

and

$$c'(z) = O(z^{p-r-1}) .$$

Together with (26) , (27) and (23) , it yields

$$\| A_n(z) \| \le C n^k / |z|^{r+1} , \| A_n'(z) \| \le C n^{k+1} / |z| \text{ for } z \in \mathcal{V} .$$

Considering (18), (22) and (23), we have proved that, for all $y \in \mathbf{R}$,

$$\| \hat{T}_n(y) \| \le C n^k \min (n, 1/|y|^{r+1}) ,$$

$$\| \hat{T}_n'(y) \| \le C n^{k+1} \min (1+n|y|^r, 1/|y|) .$$

The lemma follows from a simple calculation .

3. The case of holomorphic semigroups.

In this section, we make the stronger assumption that A generates a holomorphic semigroup on X ; more precisely we also assume that the spectrum of A is included in the sector S_θ and

$$\forall z \in \mathbf{C} \setminus S_\theta , \quad \| (zI - A)^{-1} \| \le C / |z| , \tag{28}$$

where $\theta \in]0 , \pi/2[$, C is a constant and

$$S_\theta = \{ z \in \mathbf{C} ; \pi - \theta \le |\operatorname{Arg} z| \le \pi \text{ or } z = 0 \} .$$

We can make weaker assumptions on the scheme ; we assume that the rational functions r_j and d_j are uniformly bounded in S_θ and satisfy

$$P (e^z ; z) = O(z^{p+1}) \tag{7}$$

and

$$e^{jz} - d_j(z) = O(z^p) , \quad j=1,\dots,q-1 ; \tag{29}$$

the last requirement is weaker than (5) . We suppose also that the method is A(θ)-stable with defect k , that is to say

for all z *belonging to* S_θ *, the roots of* $P(. ; z)$ *lie in the unit disk and the mulplicities of the unimodular roots are less than* k+1.

We need also the following hypothesis : there exist $\eta > 0$ and μ , $0<\mu<\cos\theta$, such that

for all root $\lambda_j(z)$ *of* $P(. ; z)$ *such that* $\lambda_j(0)$ *is unimodular and of multiplicity* k+1 *, we have* (30)
$\forall z \in S_\theta$ *with* $|z| \le \eta$, $|\lambda_j(z)| \le e^{-\mu|z|}$.

We have the following theorems :

Theorem 7. *If the scheme is A(θ)-stable with defect* k *, if conditions* (7) ,(28), (29) *and* (30) *are satisfied and if* u_0 *belongs to* $D(A^p)$ *, then the following inequality holds*

$$\| u(t_n) - u_n \|_X \le C\, t_n^k\, \Delta t^{p-k} \| A^p u_0 \|_X \ .$$

Theorem 8. *If the same assumptions are satisfied but* u_0 *belongs to* X *, if furthermore, for all* $z \in S_\theta - \{0\}$ *and for* $z = \infty$ *, the roots of* $P(\ .\ ; z)$ *lie in the open unit disk, then the following inequality holds*

$$\| u(t_n) - u_n \|_X \le C\, \Delta t^{p-k} / t_n^{p-k} \| u_0 \|_X \ .$$

Proof of Theorem 7. We use the same representation formula (8)

$$U_n - V_n = H_n(\Delta t A)\, u_0 + G_n(\Delta t A)\, u_0 \ , \qquad (8)$$

as for Theorem 1 , but instead of (11), we use the functions

$$\tilde{H}_n(z) = H_n(z)/z^p \quad , \quad \tilde{G}_n(z) = G_n(z)/z^p \ . \qquad (31)$$

Then

$$U_n - V_n = \Delta t^p \left(\tilde{H}_n(\Delta t A) + \tilde{G}_n(\Delta t A) \right) A^p u_0$$

and

$$\| U_n - V_n \|_{X^q} \le C\, \Delta t^p \left(\| \tilde{H}_n(\Delta t A) \| + \| \tilde{G}_n(\Delta t A) \| \right) \| A^p u_0 \|_X \qquad (32)$$

Since $\| \tilde{H}_n(z) \| \le C_n \min (|z| , |z|^{-p})$, we can use the Dunford-Taylor spectral representation

$$\tilde{H}_n(\Delta t A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} (z I - \Delta t A)^{-1} \tilde{H}_n(z)\, dz \ , \qquad (33)$$

where Γ_θ denotes the oriented boundary of S_θ .

From Lemma 2 and (30), we deduce that

$$\forall\ z \in \Gamma_\theta \ , \qquad \| R(z)^n \| \le C\, n^k \ ,$$

and

$$\forall\ z \in \Gamma_\theta \text{ with } |z| \le \eta \ , \quad \| R(z)^n \| \le C\, n^k\, e^{-\mu n |z|} \ ;$$

therefore

$$\forall\ z \in \Gamma_\theta \text{ with } z \ne 0 \ , \quad \| \tilde{H}_n(z) \| \le C\, n^k\, |z|^{-p}$$

and, since from (9) , $\tilde{H}_n(z) = \sum_{j=0}^{n-1} e^{jz} R(z)^{n-j-1} \tilde{H}_1(z) = O(z^{p+1})$.

$$\forall\ z \in \Gamma_\theta \text{ with } |z| \le \eta \ , \quad \| \tilde{H}_n(z) \| \le C\, n^{k+1}\, e^{-\mu n |z|}\, |z| \qquad (34)$$

Then, using (33) and (28),

$$\| \tilde{H}_n(\Delta t A) \| \le \frac{1}{\pi} \int_0^{\infty} \frac{C}{r} \| \tilde{H}_n(r e^{i\theta}) \| \, dr ,$$

$$\le C \int_0^{\eta} n^{k+1} e^{-\mu n r} \, dr + C n^k \int_{\eta}^{\infty} r^{-p-1} \, dr ,$$

$$\le C n^k .$$

We cannot use directly the Dunford-Taylor spectral representation for $\tilde{G}_n(\Delta t A)$ when $\tilde{G}_n(0) \neq 0$, but we can write

$$\tilde{G}_n(\Delta t A) = e^{n \Delta t A} \tilde{G}_n(0) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (z I - \Delta t A)^{-1} (\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)) \, dz ;$$

therefore

$$\| \tilde{G}_n(\Delta t A) \| \le C n^k + \frac{1}{\pi} \int_0^{\infty} \frac{C}{r} \| \tilde{G}_n(r e^{i\theta}) - \exp(n r e^{i\theta}) \tilde{G}_n(0) \| \, dr ,$$

and we obtain easily

$$\forall \, z \in \Gamma_\theta \text{ with } |z| \ge \eta , \quad \| \tilde{G}_n(z) - e^{nz} \tilde{G}_n(0) \| \le C n^k |z|^{-p} ,$$

and

$$\forall \, z \in \Gamma_\theta \text{ with } |z| \le \eta , \quad \| \tilde{G}_n(z) - e^{nz} \tilde{G}_n(0) \| \le C n^{k+1} e^{-\mu n |z|} |z| .$$

Thus, $\| \tilde{G}_n(\Delta t A) \| \le C n^k$ and Theorem 7 follows from (32).

Proof of Theorem 8. From the representation formula (8), we have

$$\| u(t_n) - u_n \|_X \le C (\| H_n(\Delta t A) \| + \| G_n(\Delta t A) \|) \| u_0 \|_X .$$

Now we write

$$H_n(\Delta t A) = (I - e^{\Delta t A}) H_n(\infty) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (z I - \Delta t A)^{-1} (H_n(z) - (1 - e^z) H_n(\infty)) \, dz .$$

From Lemma 2, there exists β, $0 < \beta < 1$, such that

$$\forall \, z \in \Gamma_\theta \text{ with } |z| \ge \eta , \quad \| R(z)^n \| \le C \beta^n ;$$

using also $R(z) - R(\infty) = O(z^{-1})$, (when $z \to \infty$), we obtain

$$\forall \, z \in \Gamma_\theta \text{ with } |z| \ge \eta , \quad \| H_n(z) - (1 - e^z) H_n(\infty) \| \le C n^{k+1} \beta^n |z|^{-1} ,$$

and from (34)

$\forall\, z \in \Gamma_\theta$ with $|z| \le \eta$, $\| H_n(z)-(1-e^z)H_n(\infty) \| \le C(n^{k+1}e^{-\mu n|z|}|z|^{p+1} + \beta^n |z|)$.
Therefore

$$\| H_n(\Delta tA) \| \le C \left(n^{k+1} \beta^n + \int_0^\eta n^{k+1} e^{-\mu n r} r^p \, dr \right),$$

$$\le C \left(n^{k+1} \beta^n + n^{k-p} \int_0^\infty e^{-\mu t} t^p \, dt \right),$$

$$\le C\, n^{k-p} = C \left(\Delta t / t_n \right)^{p-k}.$$

Similarly we get

$\forall\, z \in \Gamma_\theta$ with $|z| \ge \eta$, $\| G_n(z)-(1-e^z)G_n(\infty) \| \le C\, n^{k+1} \beta^n |z|^{-1}$,

and

$\forall\, z \in \Gamma_\theta$ with $|z| \le \eta$, $\| G_n(z)-(1-e^z)G_n(\infty) \| \le C(n^k e^{-\mu n|z|}|z|^p + \beta^n |z|)$,

which yields

$$\| G_n(\Delta tA) \| \le C \left(\Delta t / t_n \right)^{p-k}$$

and completes the proof of Theorem 8.

4. Remarks.

Theorem 1 is valid in particular when X is a Hilbert space but in this case it may be improved by taking Condition (29) in the place of Condition (5). Furthermore an easier proof can be given ; indeed, changing possibly the norm in X, we can assume that e^{tA} is a semigroup of contraction on X , i.e. $\| e^{tA} \| \le 1$, $(\forall\, t > 0)$; then, from a theorem of von Neumann, we have

$$\| \tilde{H}_n(\Delta tA) \| \le C \max_{\mathrm{Re}\, z = 0} \| \tilde{H}_n(z) \| \le C\, n^{k+1} ,$$

similarly , $\| \tilde{G}_n(\Delta tA) \| \le C\, n^k$, which completes the proof.

Now, if we assume further that A is a (semidefinite negative) selfadjoint operator, Theorem 7 and Theorem 8 are valid with $\theta = 0$. In this case, the proof becomes very simple since, from the spectral theory,

$$\| H_n(\Delta tA) \| \le \sup_{x \le 0} \| H_n(x) \| \ , \quad \| G_n(\Delta tA) \| \le \sup_{x \le 0} \| G_n(x) \| .$$

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